A brief tutorial on Gomory mixed integer (GMI) cuts applied to pure integer programs

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In an earlier post, I gave a brief tutorial on Gomory fractional cuts. However, Gomory fractional cuts are not used in practice. A primary reason is that they are subsumed by the more general and stronger Gomory mixed integer (GMI) cuts which were introduced in 1960 by Ralph Gomory.

Generality. GMI cuts apply when the problem has a mix of integer and continuous variables (MIPs), whereas Gomory fractional cuts only apply for problems in which all variables are integer (pure IPs).

Strength. When GMI cuts are applied to pure integer programs, they are just as strong or stronger than Gomory fractional cuts.

For simplicity, I will stick with GMI cuts as applied to pure IPs. The interested reader can consult the longer tutorial by Cornuéjols for the full version.

1 The GMI cut for pure integer programs

Suppose that nonnegative integers $x_1, \dots, x_n$ satisfy the equation $\sum_{i=1}^n a_i x_i = b$, where b is fractional. Think of this equation as a row of the simplex tableau/dictionary. Letting $I = \{1, \dots, n\}$, the associated GMI cut is:

$$\sum_{i \in I : f_i \leq f} \frac{f_i}{f} x_i + \sum_{i \in I : f_i > f} \frac{1 - f_i}{1 - f} x_i \geq 1.$$

This inequality uses the "fractional" parts of b and a_i, which are denoted $f := b - \lfloor b \rfloor$ and $f_i := a_i - \lfloor a_i \rfloor$. Each of these are nonnegative.

Notice that if $f_i \leq f$ for every i, then the resulting inequality is exactly the same as the Gomory fractional cut, which can be written as:

$$\sum_{i \in I} f_i x_i \geq f.$$

If at least one i satisfies $f_i > f$, then the GMI cut is stronger. This is because if $f_i > f$, then $\frac{1-f_i}{1-f} < \frac{f_i}{f}$, meaning the coefficient of x_i will be smaller.

2 Example from CCZ textbook

Consider the following IP.

$$\begin{aligned}
\max \quad & 5.5x_1 + 2.1x_2 \\
& -x_1 + x_2 + x_3 = 2 \\
& 8x_1 + 2x_2 + x_4 = 17 \\
& x_1,\ x_2,\ x_3,\ x_4 \geq 0 \\
& x_1,\ x_2,\ x_3,\ x_4 \text{ integer.}
\end{aligned}$$

Solving the LP relaxation gives the following system (with objective z).

$$\begin{aligned}
z \qquad\qquad & + 0.58x_3 + 0.76x_4 = 14.08 \\
x_2 & + 0.8x_3 + 0.1x_4 = 3.3 \\
x_1 \qquad & - 0.2x_3 + 0.1x_4 = 1.3.
\end{aligned}$$

This corresponds to the fractional point $(x_1, x_2, x_3, x_4) = (1.3, 3.3, 0, 0)$.

If we apply the GMI formula for row 2 of this system, we have $f = 0.3$, $f_1 = 0$, $f_2 = 0$, $f_3 = 0.8$, $f_4 = 0.1$, giving the inequality $\frac{1-0.8}{1-0.3}x_3 + \frac{0.1}{0.3}x_4 \geq 1$, or equivalently:

$$6x_3 + 7x_4 \geq 21.$$

Compare this to the (weaker) Gomory fractional cut $0.8x_3 + 0.1x_4 \geq 0.3$, or:

$$56x_3 + 7x_4 \geq 21.$$

The last row of the tableau happens to give the same GMI cut.

Unfortunately, I don't have intuitive explanations for these GMI cuts like I had for the Gomory fractional cuts in the last post. So, let's settle for a proof.

3 Proof that the GMI cut is valid

Theorem 1. *Consider the following set S, where $b \notin \mathbb{Z}$ and $I := [n]$.*

$$S = \left\{ x \in \mathbb{Z}^n_+ \;\middle|\; \sum_{i \in I} a_i x_i = b \right\},$$

Let $f := b - \lfloor b \rfloor > 0$ and $f_i := a_i - \lfloor a_i \rfloor$ for $i \in [n]$. The following GMI inequality is valid for S.

$$\sum_{i \in I : f_i \leq f} \frac{f_i}{f} x_i + \sum_{i \in I : f_i > f} \frac{1 - f_i}{1 - f} x_i \geq 1. \tag{1}$$

Proof. We show that every $x^* \in S$ satisfies inequality (1). Consider some $x^* \in S$. This implies that $x^* \in \mathbb{Z}_+^n$ and

$$\sum_{i\in I} a_i x_i^* = b. \tag{2}$$

Since $\lfloor a_i \rfloor$ and $\lfloor b \rfloor$ and x_i^* are integers, and by equation 2, we can write

$$\sum_{i\in I: f_i \le f} (a_i - \lfloor a_i \rfloor) x_i^* + \sum_{i\in I: f_i > f} (a_i - \lfloor a_i \rfloor - 1) x_i^* = b - \lfloor b \rfloor + k$$

for some integer k. In terms of our f notation, this is

$$\sum_{i\in I: f_i \le f} f_i x_i^* + \sum_{i\in I: f_i > f} (f_i - 1) x_i^* = f + k. \tag{3}$$

In the first case, suppose that $k \geq 0$, in which case

$$\begin{aligned}
\sum_{i\in I: f_i \le f} \frac{f_i}{f} x_i^* + \sum_{i\in I: f_i > f} \frac{1-f_i}{1-f} x_i^* &\geq \sum_{i\in I: f_i \le f} \frac{f_i}{f} x_i^* + 0 \\
&\geq \sum_{i\in I: f_i \le f} \frac{f_i}{f} x_i^* + \sum_{i\in I: f_i > f} \frac{f_i - 1}{f} x_i^* \\
&= \frac{f+k}{f} \geq 1.
\end{aligned}$$

The last equation holds by (3). The last inequality holds by $k \geq 0$ and $f > 0$.

In the other case, suppose $k < 0$. Then, $k \leq -1$ since k is an integer, so

$$\begin{aligned}
\sum_{i\in I: f_i \le f} \frac{f_i}{f} x_i^* + \sum_{i\in I: f_i > f} \frac{1-f_i}{1-f} x_i^* &= \sum_{i\in I: f_i \le f} \frac{f_i}{f} x_i^* + \sum_{i\in I: f_i > f} \frac{f_i - 1}{f-1} x_i^* \\
&\geq 0 + \sum_{i\in I: f_i > f} \frac{f_i - 1}{f-1} x_i^* \\
&\geq \sum_{i\in I: f_i \le f} \frac{f_i}{f-1} x_i^* + \sum_{i\in I: f_i > f} \frac{f_i - 1}{f-1} x_i^* \\
&= \frac{1}{f-1}(f+k) \geq 1.
\end{aligned}$$

The last equation holds by (3). The last inequality holds by $k \leq -1$ and $f - 1 < 0$.

So, x^* satisfies the GMI inequality (1) in both cases, so the GMI inequality is valid for S. □